HomeWork 7 : The weakly interacting Bose gas

The theory of the weakly interacting Bose gas was originally developed by Bogoliubov in the late 1940s. It was meant to be a theory of superfluid helium, although for ^{4}He the interatomic interactions are very strong. In this case the theory has some qualitative features which agree with experimental properties of ^{4}He, most notably the linear phonon like quasiparticle excitation spectrum, $\epsilon_k = ck$, at small wave vectors. But it fails to reproduce other important phenomena, such as the roton minimum in the spectrum. On the other hand, the theory is believed to give a good description of atomic BEC, since the conditions under which it is derived are close to the experimental ones.

1. Mean-field approach, the Gross-Pitaevskii equation

Suppose that the many particle state is a coherent state (ie : eigenstate of the annihilation operator) :

$$\hat{\psi}(r)|\psi\rangle = \psi_0(r)|\psi\rangle$$

The Hamiltonian of the interacting Bose gas is :

$$\begin{aligned} H &= \int \hat{\psi}^\dagger(r)\left(-\frac{\hbar^2\nabla^2}{2m} + V_1(r)\right)\hat{\psi}(r)d^3r \\ &+\frac{1}{2}\int V(r-r')\hat{\psi}^\dagger(r)\hat{\psi}^\dagger(r')\hat{\psi}(r)\hat{\psi}(r')d^3rd^3r' \end{aligned}$$

here $V_1(r)$ is an external potential.

Using $|\psi\rangle$ as a trial function, find that the dynamics of the system is ruled by the following equation :

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V_1(r) + V_{eff}(r) - \mu\right)\psi_0(r) = 0$$

Hint : apply a variational principle like we did in Homework 6 ... and do not forget the normalization of ψ_0 that will be chosen as $N_0 = \int |\psi_0(r)|^2 d^3r$;

$V_{eff}(r) = \int V(r-r')|\psi_0(r')|^2 d^3r'$

The equation so derived is called the Gross-Pitaevskii equation.

2. Beyond mean-field, the Bogoliubov transformation

Up to second order in perturbation theory and in the simple case of no external potential ($V_1(r) = 0$), the Hamiltonian of the Bose liquid becomes :

$$H_2 = \sum_k \left(\frac{\hbar^2k^2}{2m} - \mu\right)a_k^\dagger a_k + \frac{n_0 g}{2}(a_k^\dagger a_{-k}^\dagger + 4a_k^\dagger a_k + a_{-k}a_k)$$

where $\mu = n_0 g$

We introduce the Bogoliubov transformation :

$$\begin{aligned} a_k &= u_k\alpha_k + v_k\alpha_{-k}^\dagger \\ a_k^\dagger &= u_k\alpha_k^\dagger + v_k\alpha_{-k}. \end{aligned}$$

where u_k and v_k are real and obey $u_k^2 - v_k^2 = 1$.

The idea is to rewrite the Hamiltonian in terms of these new operators and then to vary the parameters u_k and v_k to make it diagonal.

a) Writing the Bogoliubov transformation in the matrix form,

$$\begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ v_k & u_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$$

show that the pair of equations can be inverted to yield

$$\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k & u_k \end{pmatrix} \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix}$$

b) Rewrite the Hamiltonian H_2 in the matrix form

$$H_2 = \sum_k \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} \epsilon_k + n_0 g & n_0 g/2 \\ n_0 g/2 & 0 \end{pmatrix} \begin{pmatrix} a_k \\ a^\dagger - k \end{pmatrix}$$

where $\epsilon_k = \hbar^2 k^2 / 2m$.

c) Use the inverse of the Bogoliubov transformation to express H_2 in terms of the $b's$ operators, namely you should find :

$$H_2 = \sum_k \begin{pmatrix} b_k^\dagger & b_{-k} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} b_k \\ b^\dagger - k \end{pmatrix}$$

where the coefficients M_{ij} have to be computed explicitly.

d) Show that the condition for the transformed matrix to be diagonal is that

$$\frac{2u_k v_k}{u_k^2 + v_k^2} = \frac{n_0 g}{\epsilon_k + n_0 g}$$

e) Show that the trace of the M matrix is

$$E = (\epsilon_k + n_0 g)(u_k^2 + v_k^2) - 2n_0 g u_k v_k.$$

Using the representation $u_k = \cosh(\theta)$ and $v_k = \sinh(\theta)$, show from d) that

$$\tanh(2\theta) = \frac{n_0 g}{\epsilon_k + n_0 g}$$

and hence prove that

$$E = \sqrt{\epsilon_k(\epsilon_k + 2n_0 g)}$$

consistent with the Bogoliubov quasiparticle dispersion given in the lecture.